Finite Math - Spring 2019

Lecture Notes - 4/25/2019

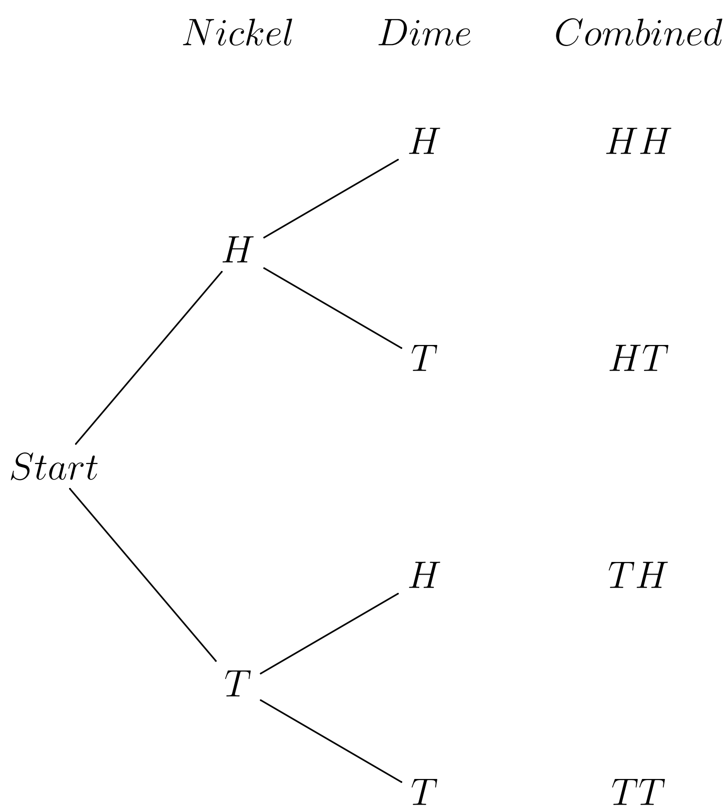
Homework

- Section 8.1 - 1, 3, 5, 7, 9, 11, 15, 17, 18, 19, 25, 26, 33, 43, 45, 47, 49, 51, 53, 63, 65, 68

Section 8.1 - Sample Spaces, Events, and Probability

Sample spaces can vary depending on what we're interested in.

Example 1. *A nickel and a dime are flipped. Depending on what we want to focus on, the sample space can vary. First, let's look at what the raw outcomes are*



(a) *If we are interested in knowing whether each coin was heads or tails, an appropriate sample space is*

since this tells us which coin came up which.

(b) *If we are only interested in the number of heads that comes up, we can choose*

as our sample space since this describes how many heads come up.

(c) *Suppose we only want to know if the coins match or not. Then we can use*

as a sample space where M stands for match and D stands for do not match.

Notice that the sample space in (a) can actually be used in the other two situations as well, and for this reason we say that S_1 is a more *fundamental sample space* than the other two.

Example 2. *An experiment consists of recording the boy-girl composition of a three-child family. What would be an appropriate sample space if:*

(a) *we are interested in the genders of the children in the order of their births? (A tree diagram can help.)*

(b) *we are interested only in the number of girls in family?*

(c) *we are interested only in which gender there are more of?*

(d) *we are interested in all three items from (a)-(c)?*

Example 3. *Suppose we run an experiment by rolling two dice. What is the most fundamental sample space for this experiment? Give the event for each of the following outcomes. Which are simple events?*

(a) *A sum of* 7 *turns up.*

(b) *A sum of* 11 *turns up.*

(c) *A sum less than* 4 *turns up.*

(d) *A sum of* 12 *turns up.*

(e) *A sum of* 5 *turns up.*

(f) *A sum which is prime turns up.*

(g) *A sum of* 2 *turns up.*

Probability of an Event.

Definition 1 (Probabilities of Simple Events)**.** *Given a sample space*

$$S = \{e_1, e_2, ..., e_n\}$$

with n simple events, to each simple event e_i we assign a real number, denoted by $P(e_i)$, called the probability of event e_i*. These numbers can be assigned in an arbitrary manner as long as the following two conditions are satisfied:*

Condition 1: *The probability of a simple event is a number between* 0 *and* 1*, inclusive, i.e.,*

$$0 \leq P(e_i) \leq 1.$$

Condition 2: *The sum of the probabilities of all simple events in the sample space is* 1*, i.e.,*

$$P(e_1) + P(e_2) + \cdots + P(e_n) = 1.$$

Any probability assignment that satisfies Conditions 1 and 2 is said to be an acceptable probability assignment*.*

A simple example to illustrate this is flipping one coin. The sample space for this would be $S = \{H, T\}$. We would assume that there's a "50-50 chance" that the coin will turn up heads or tails, so it seems reasonable to assign the following probabilities:

$$P(H) = \frac{1}{2} \qquad P(T) = \frac{1}{2}.$$

Notice that this satisfies the two conditions of the previous definition. Is this a reasonable assignment of probabilities? Ostensibly yes since there are only two outcomes of a coin flip and there is no reason to doubt that the coin is *fair*, i.e., the two outcomes are equally likely.

However, suppose we flipped the coin 1000 times and we turn up with 373 heads and 627 tails. This could suggest that the coin is somehow "rigged" to land on tails more often. In this case, we might consider assigning the probabilities as

$$P(H) = \frac{373}{1000} = 0.373 \qquad P(T) = \frac{627}{1000} = 0.627$$

since that reflects the actual results of a large collection of outcomes.

Another assignment we could technically make is

$$P(H) = 1 \qquad P(T) = 0.$$

While this does fit the rules of an acceptable probability assignment, it is not *reasonable* in this case, unless the coin had two heads.

An example of an unacceptable probability assignment is

$$P(H) = 0.6 \qquad P(T) = 0.8$$

since $P(H) + P(T) = 1.4 > 1$.

Definition 2 (Probability of an Event E)**.** *Given an acceptable probability assignment for the simple events in a sample space S, we define the* probability of an arbitrary event E*, denoted $P(E)$, as follows:*

(a) *If E is the empty set, then $P(E) = 0$.*

(b) *If E is a simple event, then $P(E)$ has already been assigned.*

(c) *If E is a compound event, then $P(E)$ is the sum of the probabilities of all the simple events in E.*

(d) *If E is the sample space S, then $P(E) = P(S) = 1$ (this is a special case of part (c).)*

Example 4. *Refer back to the example where we roll two dice. If we assume that every simple event is equally likely:*

(a) *What is the probability of a simple event happening?*

(b) *What are the possible numbers that the two dice could add up to?*

(c) *What are the probability of each of the events in part (b) happening?*

Equally Likely Assumption. When we were talking about assigning probabilities of 0.5 to heads and 0.5 to tails for flipping a coin, and a probability of $\frac{1}{6}$ for any number to come up when rolling a 6-sided die, we are making an assumption on the probabilities of the experiment called an *equally likely assumption.* In an ideal case, this poses no risk, but as we talked about, a coin or a die may not necessarily be "fair."

Generally, if the sample space is

$$S = \{e_1, e_2, ..., e_n\},$$

we assign to each e_i a probability of $\frac{1}{n}$ since there are n possible outcomes and we want each of them to be equally likely. This gives us the following theorem

Theorem 1 (Probability of an Arbitrary Event under an Equally Likely Assumption)**.** *If we assume that each simple event in a sample space S is equally likely to occur, then the probability of an arbitrary event E in S is given by*

$$P(E) = \frac{n(E)}{n(S)},$$

the number of elements in E divided by the number of elements in S.

We saw this theorem in action when we found the theoretical probabilities for rolling a number on a pair of dice.

Empirical Approach. In an empirical approach to probability, we run the experiment several times, and assign probabilities according to the frequency which with outcomes occurred. For example, if we flip a coin 1000 times and get 373 heads and 627 tails, we would be tempted to assign probabilities as

$$P(H) = \frac{373}{1000} \qquad P(T) = \frac{623}{1000}$$

since it reflects the results of an extensive experiment.

The number of times an event E occurs in an experiment is called the *frequency* of the event, and is denoted $f(E)$. If the experiment has n trials (in the example above, $n = 1000$ since there was 1000 coin flips), the *relative frequency* of the event E in n trials is the number $\frac{f(E)}{n}$. We can define the *empirical probability* of E, which we will denote by $P(E)$, by the number that $\frac{f(E)}{n}$ approaches as n gets larger and larger. The reasoning behind this is that given an infinite number of runs of the experiment, the relative frequency should reflect the actual probabilities, and generally, this number becomes a better and better approximation as the number of runs of the experiment increases. For any finite number n we plug in, $\frac{f(E)}{n}$ is an *approximate empirical probability* of the event E.